AIAA JOURNAL
Vol. 34, No. 9, September 1996

Cross-Sectional Analysis of Composite Beams Including Large Initial Twist and Curvature Effects

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An asymptotically exact methodology, based on geometrically nonlinear, three-dimensional elasticity, is presented for cross-sectional analysis of initially curved and twisted, nonhomogeneous, anisotropic beams. The analysis is subject only to the restrictions that the strain is small relative to unity and that the maximum dimension of the cross section is small relative to the wavelength of the deformation and to the minimum radius of curvature and/or twist. The final one-dimensional strain energy per unit length exhibits asymptotically correct second-order dependence of the initial curvature and twist parameters. Cross-sectional constants of the one-dimensional theory are obtained via finite element discretization over the cross-sectional plane. Numerical results obtained for both isotropic and composite beams are compared with published results from special purpose analyses for initially twisted, straight beams, as well as initially curved, untwisted beams. The agreement with previously published results is excellent.

Introduction

BECAUSE of their geometries, rotor blades, wing structural boxes, and many other engineering structures have one dimension that is much larger than the other two. Such flexible structures can often be treated as a beam, a one-dimensional body. This idealization of the actual structure leads to a much simpler mathematical formulation than would be obtained if complete three-dimensional elasticity were used to model it. To do so, one has to find a way to capture the behavior associated with the two dimensions that are being eliminated by correct accounting for geometry and material distributions. The process that takes the original three-dimensional body and represents it as one-dimensional is called dimensional reduction.

Although dimensional reduction processes can be extremely simple for homogeneous, isotropic, prismatic beams, and especially for restricted cases of deformation, they are far less tractable for composite beams undergoing arbitrary deformation. Specifically, difficulties arise in obtaining a one-dimensional strain energy function that is equivalent, at least in some sense, to a three-dimensional representation. For anisotropic beams, all possible deformations of the three-dimensional structure must be included in the formulation, as described, among others, by Refs. 1–3.

Many investigators have approached the problem of naturally twisted and curved beams in the literature, although no one has carried this out in a general way and in such a suitable framework as is proposed in this research. Most of the existing analyses invoke assumptions that restrict the model from being used just for certain types of cross sections and/or deformations.[4,5] Also, most sectional models do not stem from the same theory as that which governs the global deformations. Even the work of Borri et al.,[3] one of the most sophisticated cross-sectional analysis, dealing with initial twist as well as initial curvature for anisotropic beams, falls in this category. A more detailed overview of works in this field can be found in Refs. 6 and 7.

Presented as Paper 95-1500 at the AIAA/ASME/ASCE/AHS/ASC 36th Structures, Structural Dynamics, and Materials Conference, New Orleans, LA, April 10–12, 1995; received July 6, 1995; revision received Sept. 28, 1995; accepted for publication Oct. 1, 1995.

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This paper continues earlier work of the authors,[8] where we presented an anisotropic beam theory from geometrically nonlinear, three-dimensional elasticity based on an asymptotical approach that, in turn, relies on the identification of small parameters. The cross section may have arbitrary geometry (solid or thin walled, closed or open). The idea is to be able to model complex structures (e.g., an actual airfoil-shaped cross section, with all its components and different materials), not just some simplified version of it. This general framework, described in detail in Ref. 9, allows us to incorporate several effects coming from the three-dimensional formulation in a consistent manner, without the use of ordering schemes or other ad hoc restrictions. One of those effects is the presence of curvature and twist in the undeformed state of the beam. The importance of including those initial effects in the geometrically exact equilibrium and kinematical equations, and in the calculation of sectional constants was shown in another paper of the first two authors,[10] where a linear correction was derived. However, there are certain applications (e.g., tilt-rotor blades, propfans) where the beam is highly twisted or curved, and a more refined theory is needed. Such a theory is presented in this paper. The limitations of the first-order correction, its comparison with the second-order one, and the possible loss of positive definiteness of the stiffness matrix are all discussed in Ref. 11 and, therefore, are not repeated here.

The developed theory was implemented numerically in a computer code called variational-asymptotical beam section analysis (VABS). From it one can get the stiffness constants and warping field over the cross section. Along with the one-dimensional code of, for example, Ref. 12 (based on the work of Ref. 13), beam results can be generated. Results from the current work were compared with other numerical and analytical results whenever available.

Cross-Sectional Analysis

In constructing a one-dimensional beam theory from three-dimensional elasticity, we attempt to represent the strain energy stored in a three-dimensional body by finding the strain energy that would be stored in an imaginary one-dimensional body. This modeling process cannot be performed in an exact manner. However, due to the interest of working with a simple one-dimensional theory, researchers have turned to asymptotical methods to reduce the dimension of the model for bodies that contain one or more small parameters.

Thus, in what follows we replace the three-dimensional beam problem by an approximate one-dimensional one in which the strain energy per unit length will be a function only of $x_1 \equiv x$ (length

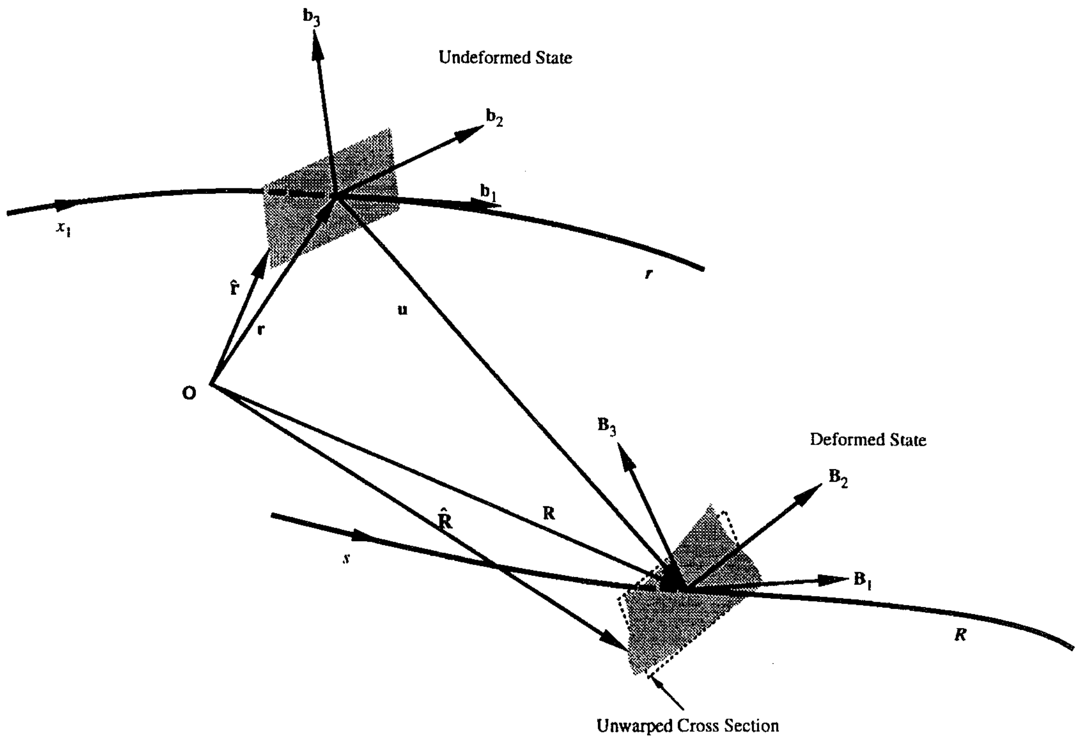

Fig. 1 Schematic of beam deformation.

along a reference line r within an undeformed beam; $x_2 \equiv y_2$ and $x_3 \equiv y_3$ denote lengths along lines orthogonal to the reference line r; see Fig. 1). This will be done with the aid of the variational-asymptotical formulation.[14,15] The kinematics of the beam are based on the general formulation of Danielson and Hodges.[16] Local rotation, as defined therein, is taken to be of the order of the strain. Since only geometrically nonlinear behavior is considered, the strain can be treated as small relative to unity without imposing any explicit restrictions on the magnitudes of the displacement of the reference line or the section rotation. Subject only to these restrictions, all possible deformations of beams are taken into account in the analysis. Even though some of the theoretical development is repeated herein for completeness, more detail can be found in Refs. 9 and 16.

Three-Dimensional Formulation

As described in Refs. 8 and 17, we first derive the three-dimensional formulation, the solution of which shall be considered the exact solution of the beam problem.

Strain Field

From the work of Danielson and Hodges,[16] under the condition of small local rotation, Jaumann strain components Γ^* (a 3×3 symmetric matrix) can be expressed by

$$\Gamma^* = \tfrac{1}{2}(\chi + \chi^T) - I$$
$$\chi_{mn} = \boldsymbol{B}_m \cdot \frac{\partial \hat{\boldsymbol{R}}}{\partial x_k} \boldsymbol{g}^k \cdot \boldsymbol{b}_n \tag{1}$$

where I is the 3×3 identity matrix, $\hat{\boldsymbol{R}}$ the position vector to an arbitrary point in the deformed beam, $\boldsymbol{g}^k$ the contravariant base vector for the undeformed state, and $\boldsymbol{B}_m$ and $\boldsymbol{b}_n$ are components of the orthonormal reference triads in the deformed and undeformed states, respectively.

From the preceding equation, one can express the strain field as a 6×1 column matrix:

$$\Gamma = \lfloor \Gamma^*_{11} \quad 2\Gamma^*_{12} \quad 2\Gamma^*_{13} \quad \Gamma^*_{22} \quad 2\Gamma^*_{23} \quad \Gamma^*_{33} \rfloor^T$$

so that

$$\Gamma = (1/h)\Gamma_h v + \Gamma_\epsilon \epsilon + \Gamma_R v + \Gamma_l v' \tag{2}$$

where matrices $\Gamma_h(6 \times 3)$, $\Gamma_\epsilon(6 \times 4)$, $\Gamma_R(6 \times 3)$, and $\Gamma_l(6 \times 3)$ are

$$\Gamma_h = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial}{\partial \zeta_2} & 0 & 0 \\ \frac{\partial}{\partial \zeta_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial \zeta_2} & 0 \\ 0 & \frac{\partial}{\partial \zeta_3} & \frac{\partial}{\partial \zeta_2} \\ 0 & 0 & \frac{\partial}{\partial \zeta_3} \end{bmatrix} \tag{3}$$

$$\Gamma_\epsilon = \frac{1}{\sqrt{g}} \begin{bmatrix} 1 & 0 & -\zeta_3 & \zeta_2 \\ 0 & \zeta_3 & 0 & 0 \\ 0 & -\zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma_R = \frac{1}{\sqrt{g}} \begin{bmatrix} \tilde{k} + k_1 I \left(\zeta_3 \frac{\partial}{\partial \zeta_2} - \zeta_2 \frac{\partial}{\partial \zeta_3} \right) \\ 0 \end{bmatrix} \qquad \Gamma_l = \frac{1}{\sqrt{g}} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

Here $\tilde{()}_{nm} = -e_{nmk}()_k$, and the column matrix ϵ represents the one-dimensional measures of deformation

$$\epsilon = \begin{Bmatrix} \gamma \\ h\kappa \end{Bmatrix} \tag{4}$$

where γ is the average cross-sectional extensional strain (the axial force strain measure) defined as

$$\gamma = \boldsymbol{R}' \cdot \boldsymbol{B} - 1 \tag{5}$$

and column matrix $\kappa = \lfloor \kappa_1 \ \ \kappa_2 \ \ \kappa_3 \rfloor^T$ contains the so-called moment strain measures

$$\kappa_n = K_n - k_n \tag{6}$$

with $\boldsymbol{k} = k_i \boldsymbol{b}_i$, where k_1 is the initial twist angle per unit length and k_2, k_3 are initial curvature components. The metric determinant g can be calculated as

$$\sqrt{g} = 1 - y_2 k_3 + y_3 k_2 \tag{7}$$

where $y \equiv \{y_2, y_3\}$ are cross-sectional local Cartesian coordinates that vary in the prescribed domain S, with area $|S|$. The characteristic size of the domain S is denoted by h and the dimensionless coordinates $\zeta \equiv \{\zeta_2 \equiv y_2/h, \zeta_3 \equiv y_3/h\}$ are introduced.

The small parameter ε can be now specified as

$$\varepsilon = \max \|\epsilon\| \tag{8}$$

A few nonlinear terms in the strain field, which couple v and ϵ, have been neglected in Eq. (2) because a physically linear beam theory is to be developed. The form of the strain field is of great importance because it is now linear in ϵ, v, and its derivatives. This is the only point where ε as a small parameter needs to be taken into account.

Strain Energy of a Beam

The strain energy density for a beam per unit length can be written as

$$U = \tfrac{1}{2}\langle \Gamma^T D \Gamma \rangle \tag{9}$$

where D is the 6×6 symmetric material matrix in the $\boldsymbol{b}_n$ basis and the notation

$$\langle \cdot \rangle = \int_S \cdot \sqrt{g}\, \mathrm{d}y_2\, \mathrm{d}y_3 = h^2 \int \cdot \sqrt{g}\, \mathrm{d}\zeta_2\, \mathrm{d}\zeta_3 \tag{10}$$

is used throughout the paper.

The three-dimensional Jaumann stress Z, which is conjugate to the Jaumann strain Γ, is

$$Z = D\Gamma \tag{11}$$

Small Parameters

There are four characteristic parameters in the considered theory, two of which, h and ε, have already been introduced. Two others are the characteristic length l over which the deformation state varies in the longitudinal direction, and the characteristic length of the initial curvature and twist $R = 1/\max \|\boldsymbol{k}\|$.

Consider the situation in which the parameters h, l, R, and ε are present. It is clear that the first term of Eq. (2) has order $\|v\|/h$, the second has order ε, the third has order $\|v\|/R$ and the last has order $\|v\|/l$. The fourth term has order h/l times that of the first. We should neglect this as a higher order term in the first approximation if we intend to expand the solution with respect to the small parameter h/l. This important circumstance will allow us to avoid the presence of derivatives of the unknown functions $v_n(x, \zeta)$ with respect to x for any approximation and then to solve it in an appropriate form. The third term has order h/R times that of the first. We should also neglect this as a higher order term in the first approximation. Note that the parameter h/R is also present inside some matrices Γ defined in Eq. (3). This may be disregarded, as we do not need an additional expansion of h/R. The parameter ε does not need to be considered small any more, since our main problem has become linear with respect to the unknown functions $v_n(x, \zeta)$ and the one-dimensional strain measure ϵ. As a small parameter, ε has already been taken into account (see subsection Strain Field).

We will expand the warping $v_n(x, \zeta)$ as a series with respect to the small parameters h/l and h/R. Since both of them have the same numerator, expansion in h/l and h/R is the same as the expansion in h only.

We can therefore consider h to be the only small parameter in spite of its dimension.

Discretization

The problem may be solved numerically by discretizing it with respect to the cross-sectional coordinates ζ_α. Considering the finite element discretization, the unknown functions $v_n(x, \zeta)$ can be represented as the product of a shape function matrix $S(\zeta_\alpha)$ and a column matrix of nodal values of $v(x, \zeta)$, denoted as $V(x)$,

$$v(x, \zeta) = S(\zeta)V(x) \tag{12}$$

Substituting the preceding discretized unknown function into Eq. (9) and also taking into account Eq. (2), one obtains

$$\begin{aligned} 2U &= (1/h)^2 V^T E V + (1/h) 2V^T (D_{h\varepsilon}\epsilon + D_{hR} V + D_{hl} V') \\ &\quad + (1)\big(\epsilon^T D_{\varepsilon\varepsilon}\epsilon + V^T D_{RR} V + V'^T D_{ll} V' \\ &\quad + 2V^T D_{R\varepsilon}\epsilon + 2V'^T D_{l\varepsilon}\epsilon + 2V^T D_{Rl} V'\big) \end{aligned} \tag{13}$$

in which the following definitions were introduced:

$$\begin{aligned} E &\triangleq \langle [\Gamma_h S]^T D [\Gamma_h S] \rangle & D_{\varepsilon\varepsilon} &\triangleq \langle [\Gamma_\varepsilon]^T D [\Gamma_\varepsilon] \rangle \\ D_{h\varepsilon} &\triangleq \langle [\Gamma_h S]^T D [\Gamma_\varepsilon] \rangle & D_{hR} &\triangleq \langle [\Gamma_h S]^T D [\Gamma_R S] \rangle \\ D_{hl} &\triangleq \langle [\Gamma_h S]^T D [\Gamma_l S] \rangle & D_{R\varepsilon} &\triangleq \langle [\Gamma_R S]^T D [\Gamma_\varepsilon] \rangle \\ D_{l\varepsilon} &\triangleq \langle [\Gamma_l S]^T D [\Gamma_\varepsilon] \rangle & D_{RR} &\triangleq \langle [\Gamma_R S]^T D [\Gamma_R S] \rangle \\ D_{Rl} &\triangleq \langle [\Gamma_R S]^T D [\Gamma_l S] \rangle & D_{ll} &\triangleq \langle [\Gamma_l S]^T D [\Gamma_l S] \rangle \end{aligned} \tag{14}$$

Warping Field for a Prismatic Beam

We reproduce part of the derivation done in Ref. 17 to provide the appropriate background for the next section. According to the variational asymptotical procedure, to get the next approximation, one should retain only the leading terms in the energy expression. These are leading terms with respect to the small parameter that contains the unknown functions and the leading intersection terms between the unknown functions and the rest of the functional (for more details see Ref. 18).

We are then left with the following expression:

$$2U = (1/h)^2 V^T E V + (1/h) 2V^T D_{h\varepsilon}\epsilon \tag{15}$$

This functional must be minimized with respect to the variable V, subject to the constraints

$$V^T H \Psi_{cl} = 0 \tag{16}$$

where

$$H \triangleq \langle S^T S \rangle \tag{17}$$

and Ψ_{cl} is a matrix with four columns, each corresponding to one of the constraints defined in Ref. 8. The set of column Ψ_{cl} is determined by the kernel (null space) of the matrix E. This implies

$$E\Psi_{cl} = 0 \tag{18}$$

Let us suppose that the set of columns of Ψ_{cl} is normalized in such a way that

$$\Psi_{cl}^T H \Psi_{cl} = I \tag{19}$$

The Euler equation for the minimization problem defined by Eqs. (15) and (16) is given by

$$(1/h)EV + D_{h\varepsilon}\epsilon = H\Psi_{cl}\mu \tag{20}$$

where μ/h is the column matrix of Lagrange multipliers associated with Eq. (16). By premultiplying Eq. (20) by Ψ_{cl}^T, one can prove that

$$\mu = \Psi_{cl}^T D_{h\varepsilon}\epsilon \tag{21}$$

Subsequently, Eq. (20) can be rewritten as

$$(1/h)EV = -\big(I - H\Psi_{cl}\Psi_{cl}^T\big) D_{h\varepsilon}\epsilon \tag{22}$$

The matrix E possesses a zero eigenvalue of multiplicity four, and thus its inverse does not exist. However, let us introduce the matrix E_{cl}^{+} with the following properties[8]:

$$E E_{cl}^{+} = I - H \Psi_{cl} \Psi_{cl}^{T}$$
$$E_{cl}^{+} E = I - \Psi_{cl} \Psi_{cl}^{T} H \tag{23}$$
$$E_{cl}^{+} E E_{cl}^{+} = E_{cl}^{+}$$

The solution of Eq. (22) is then represented by

$$V = -h E_{cl}^{+} D_{h\varepsilon} \epsilon \triangleq h V_0 \tag{24}$$

Note that by substituting the preceding solution into the discretized strain energy density, Eq. (13), and keeping only terms with the lowest order, which are equal to $h^0 \equiv 1$, one obtains

$$2U = \epsilon^T A_{cl}^{0} \epsilon \tag{25}$$

with

$$A_{cl}^{0} \triangleq D_{\varepsilon\varepsilon} - [D_{h\varepsilon}]^T E_{cl}^{+} [D_{h\varepsilon}] \tag{26}$$

which is the classical result for beam energy. Note that the third property from Eq. (23) is taken into account.

Higher-Order Corrections to the Warping Field

Since the warping field is function of the small parameter h, it can be expanded in the following manner:

$$V = h V_0 + h^2 V_1 + h^3 V_2 \tag{27}$$

where V_1, and V_2 are first- and second-order perturbations in the warping field, respectively.

Substituting Eq. (27) into Eq. (13), and keeping all of the terms up to order h^2, one gets

$$\begin{aligned} 2U = {} & (1)\left(\epsilon^T D_{\varepsilon\varepsilon}\epsilon + V_0^T E V_0 + 2 V_0^T D_{h\varepsilon}\epsilon\right) \\ & + 2(h)\left(\underline{V_0^T E V_1} + \underline{V_1^T D_{h\varepsilon}\epsilon} + V_0^T D_{hR} V_0 + V_0^T D_{R\varepsilon}\epsilon\right) \\ & + (h)^2\Big[\underline{\underline{2 V_0^T E V_2}} + V_1^T E V_1 + 2 V_0^T \left(D_{hR} + D_{hR}^T\right) V_1 \\ & + \underline{\underline{2 V_2^T D_{h\varepsilon}\epsilon}} + V_0^T D_{RR} V_0 + 2 V_1^T D_{R\varepsilon}\epsilon\Big] \end{aligned} \tag{28}$$

The underlined terms, as well as the double-underlined ones, cancel out each other due to Eqs. (16) and (22). This means that V_2 is not needed to get the energy asymptotically correct up to order h^2. The term V_1 is calculated next.

Warping Correction Due to Initial Twist/Curvature

As previously shown, only one step further in the warping correction is needed to include quadratic effects of initial twist and curvature in the strain energy. Therefore, consider the following substitution:

$$V = h V_0 + h^2 V_1 \tag{29}$$

where $h V_0$ is given by Eq. (24), and V_1 is an unknown function. Substituting Eq. (29) into Eq. (13), and keeping only the leading terms, one gets

$$2U = (h)^2\left[V_1^T E V_1 + V_0^T\left(D_{hR} + D_{hR}^T\right) V_1 + V_1^T D_{R\varepsilon}\epsilon\right] \tag{30}$$

The Euler equation for the minimization problem defined by Eqs. (30) and (16) is given by

$$E V_1 + \left(D_{hR} + D_{hR}^T\right) V_0 + D_{R\varepsilon}\epsilon = H \Psi_{cl} \mu \tag{31}$$

where μ is the column matrix of Lagrange multipliers associated with the constraints. In the same way done in the prismatic case, one can isolate μ by premultiplying Eq. (31) by Ψ_{cl}^T:

$$\mu = \Psi_{cl}^T\left(D_{hR} + D_{hR}^T\right) V_0 + \Psi_{cl}^T D_{R\varepsilon}\epsilon \tag{32}$$

Therefore, Eq. (31) can be rewritten as

$$E V_1 = -\left(I - H \Psi_{cl} \Psi_{cl}^T\right)\left[\left(D_{hR} + D_{hR}^T\right) V_0 + D_{R\varepsilon}\epsilon\right] \tag{33}$$

The final solution for V_1 is given by

$$V_1 = -E_{cl}^{+}\left[D_{R\varepsilon} - \left(D_{hR} + D_{hR}^T\right) E_{cl}^{+} D_{h\varepsilon}\right]\epsilon \tag{34}$$

Stiffness Matrix

The final form of the classical strain energy with corrections of order h^2/R^2 can be obtained by substituting Eqs. (24) and (34) into Eq. (28), leading to

$$2U = \epsilon^T A_{cl}^{rr} \epsilon \tag{35}$$

where the stiffness matrix is

$$\begin{aligned} A_{cl}^{rr} \triangleq {} & (1)\left[D_{\varepsilon\varepsilon} - (D_{h\varepsilon})^T\left(E_{cl}^{+} D_{h\varepsilon}\right)\right] + h\Big[\left(E_{cl}^{+} D_{h\varepsilon}\right)^T \\ & \times \left(D_{hR} + D_{hR}^T\right)\left(E_{cl}^{+} D_{h\varepsilon}\right) - \left(E_{cl}^{+} D_{h\varepsilon}\right)^T D_{R\varepsilon} - D_{R\varepsilon}^T\left(E_{cl}^{+} D_{h\varepsilon}\right)\Big] \\ & + h^2\Big\{\left(E_{cl}^{+} D_{h\varepsilon}\right)^T D_{RR}\left(E_{cl}^{+} D_{h\varepsilon}\right) - \Big[D_{R\varepsilon} - \left(D_{hR} + D_{hR}^T\right) \\ & \times E_{cl}^{+} D_{h\varepsilon}\Big]^T E_{cl}^{+}\left[D_{R\varepsilon} - \left(D_{hR} + D_{hR}^T\right) E_{cl}^{+} D_{h\varepsilon}\right]\Big\} \end{aligned} \tag{36}$$

One can see from the result in Eq. (36) that the prismatic solution coincides with the classical solution for composite beams.[8] Two levels of correction due to effects of initial curvature and twist are provided, and the full equation should be used when $h/R > 0.1$ or when the total energy error[10] becomes unacceptable.

Numerical Results

Although the theory was derived for cross sections of general geometry, the examples presented here will be restricted to some simple configurations. We can concentrate our attention on the performance and accuracy of the present theory in comparison with others available from the literature. For a realistic multicomponent cross-sectional configuration, see, for example, Ref. 19, where an actual rotor blade cross section is analyzed used VABS.

The present theory is exercised for an isotropic test beam, as well as for geometrically similar composite beams with two different layups. In what follows, A_{cl}^{rr} represents the stiffness matrix (1 extension; 2 torsion; 3 and 4 bending), and the inverse of it is the flexibility matrix, represented by F_{cl}^{rr}.

Isotropic Test

Consider a beam with rectangular cross section with dimensions $2a_2 = 26.8$ mm and $2a_3 = 2.68$ mm, where a_2 and a_3 are half-width and half-thickness, respectively, discretized with a 12×40 six-node-element mesh (quadratic interpolation across the width). The material has properties $E = 26$ MPa and $\nu = 0.3$. This test case allows us to compare the pretwist results against those for the helicoidal shells of Knowles and Reissner[20] and the beam analyses of Chu[21] and Rosen.[22] For the initially curved configuration, the comparisons are limited to the asymptotical theory of Berdichevsky and Staroselsky.[23]

Initially Twisted Case

As discussed in Ref. 20, not only an extension-twist coupling arises in the presence of pretwist but also a change in the extension and bending stiffnesses of a thin strip. The closed-form solution in the form of a series expansion for the effective extensional rigidity ratio ($1/F_{cl(1,1)}^{rr}$), effective torsional rigidity ratio ($1/F_{cl(2,2)}^{rr}$), and the extension-twist stiffness coupling ratio

$$\frac{A_{cl(1,2)}^{rr}}{EI_{22} + EI_{33}}$$

up to cubic terms in k_1 are given by[20]

$$\left.\frac{1}{EAF_{cl(1,1)}^{rr}}\right|_{K\&R} = \frac{1 - \frac{7}{6}a_2^2 k_1^2 + \left(Ea_2^4/15Ga_3^2\right)k_1^2}{1 - [(2-2\nu)/3]a_2^2 k_1^2 + \left(3Ea_2^4/20Ga_3^2\right)k_1^2} \tag{37}$$

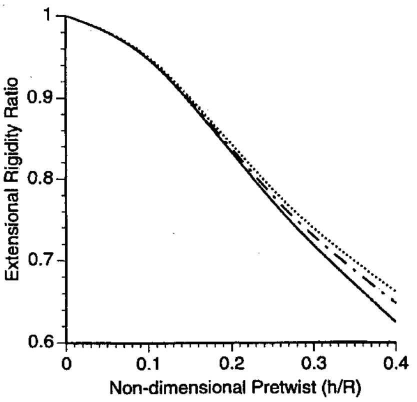

Fig. 2 Reduction in the effective extensional rigidity ratio due to initial twist for a thin strip isotropic beam: ——, VABS; · · · ·, Chu[21]; and – - -, K&R–quadratic.

$$\left.\frac{1}{GJF^{rr}_{cl(2,2)}}\right|_{K\&R} = 1 - \frac{7}{6}a_2^2 k_1 + \frac{1}{15}\frac{E}{G}\frac{a_2^4}{a_3^2}k_1^2 \tag{38}$$

$$\left.\frac{A^{rr}_{cl(1,2)}}{EI_{22} + EI_{33}}\right|_{K\&R} = \left(1 - \frac{9+8\nu}{10}a_2^2 k_1^2\right)k_1 \tag{39}$$

where EA and GJ are the extensional and torsional stiffnesses for a prismatic beam, respectively, E is the Young's modulus, ν is the Poisson's ratio, G is the shear modulus, and $K\&R$ stands for Knowles and Reissner.[20]

The effective extensional rigidity ratio is also given in Ref. 20 in a form that is analogous to the approximation of Ref. 21:

$$\left.\frac{1}{EAF^{rr}_{cl(1,1)}}\right|_{C} = \frac{1 + \left(Ea_2^4/15Ga_3^2\right)k_1^2}{1 + \left(3Ea_2^4/20Ga_3^2\right)k_1^2} \tag{40}$$

where C stands for Chu's approximation.

Under the assumption of a thin cross section, Ref. 22 coincides with the result of Ref. 21 for the effective torsional rigidity ratio, given by

$$\left.\frac{1}{GJF^{rr}_{cl(2,2)}}\right|_{R} = 1 + \frac{1}{15}\frac{E}{G}\frac{a_2^4}{a_3^2}k_1^2 \tag{41}$$

where R stands for Rosen.[22]

Finally, from the asymptotical work of Berdichevsky and Staroselsky,[23] the extension-twist stiffness coupling ratio is given by

$$\left.\frac{A^{rr}_{cl(1,2)}}{E(I_{22} + I_{33})}\right|_{B\&S} = \left(1 - \frac{J}{I_{22} + I_{33}}\right)k_1 \tag{42}$$

where $I_{\alpha\alpha}$ are the area moments of inertia, J is the torsional constant, and $B\&S$ stands for Berdichevsky and Staroselsky.[23]

Figures 2–4 show the results from VABS using the second-order correction in h/R in the stiffness matrix and the aforementioned theories. The suffix added to the results from $K\&R$ are related to the highest order of k_1 kept on the series expansions from Eqs. (37–39). As one can see, very good agreement is shown among all of the results, as expected. Note that if VABS were run only for the first-order-correction stiffness matrix in the isotropic case, the same results would be found for the coupling term (linear variation), but there would be no change in either extension or torsional stiffness constants (even though a variation in the corresponding effective rigidities would be exhibited due to the coupling term).

Initially Curved Case

For the presence of initial curvature, we are going to restrict our comparisons to Ref. 23, because of a lack of published stiffness results for this case. Further validation studies could be conducted in the one-dimensional domain, where several results are available for vibration of curved beam (e.g., Ref. 24). These will be presented in a later paper.

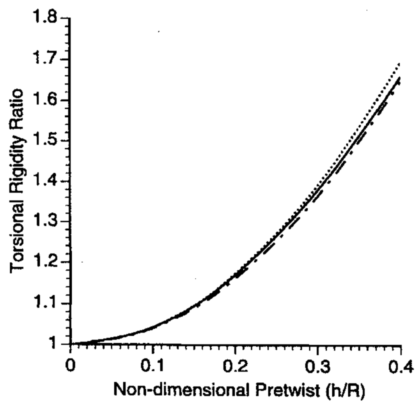

Fig. 3 Increase in the effective torsional rigidity ratio due to initial twist for a thin-strip isotropic beam: ——, VABS; · · · ·, Rosen,[22] Chu[21]; and – - -, K&R–quadratic.

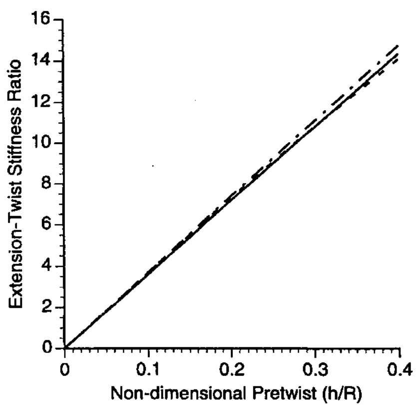

Fig. 4 Extension-twist stiffness coupling ratio due to initial twist for a thin-strip isotropic beam: ——, VABS, B&S; - - -, K&R–linear; and – - -, K&R–cubic.

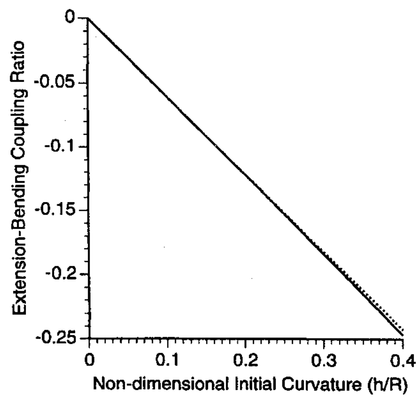

Fig. 5 Extension-bending stiffness coupling ratio $[A^{rr}_{cl(1,4)}/EI_{33}]$ due to initial curvature (k_3) for a thin-strip isotropic beam: ——, VABS and · · · ·, B&S.

Since the asymptotical work of Berdichevsky and Staroselsky[23] deals with the first-order approximation for the stiffness matrix, there is the presence of extension-bending coupling only. From that, the coupling terms are

$$\left.\frac{A^{rr}_{cl(1,3)}}{EI_{22}}\right|_{B\&S} = -(1+\nu)k_2$$
$$\left.\frac{A^{rr}_{cl(1,4)}}{EI_{33}}\right|_{B\&S} = -(1+\nu)k_3 \tag{43}$$

The present linear formulation was tested before[10] based on a square cross section. The agreement between the numerical and closed-form [Eq. (43)] solutions was excellent. For the thin-strip case, Fig. 5 shows the comparision between VABS and Eq. (43) for the case where one has an initial curvature (k_3) along the flatwise

Table 1 Properties of T300/5208 graphite/epoxy unidirectional fibers (L direction is along the fibers and N is normal to the laminate)

$E_{LL} = 132.2$ GPa	$E_{TT} = E_{NN} = 10.75$ GPa
$G_{LT} = G_{LN} = G_{TN} = 5.65$ GPa	
$\nu_{LT} = \nu_{LN} = 0.239$	$\nu_{TN} = 0.400$

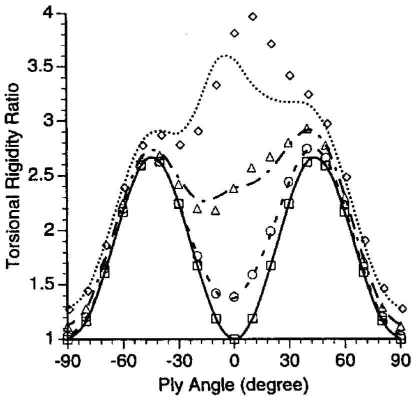

Fig. 6 Torsional rigidity ratio of a $[\theta_2/-\theta_2]_T$ composite thin-strip beam for different levels of initial twist: ——, VABS − h/R = 0; - - -, VABS − h/R = 0.1; — -, VABS − h/R = 0.2; ...., VABS − h/R = 0.3; □, $K - h/R = 0$; ○, $K - h/R = 0.1$; △, $K - h/R = 0.2$; and ◇, $K - h/R = 0.3$.

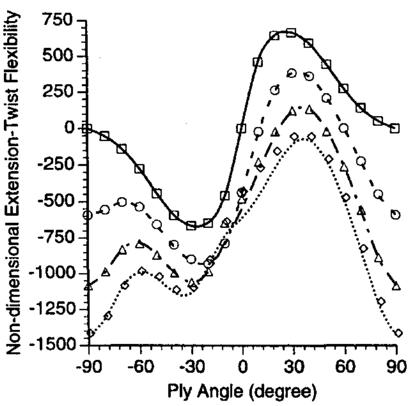

Fig. 7 Extension-twist coupling flexibility of a $[\theta_2/-\theta_2]_T$ composite thin strip beam for different levels of initial twist: ——, VABS − h/R = 0; - - -, VABS − h/R = 0.1; — -, VABS − h/R = 0.2; ...., VABS − h/R = 0.3; □, $K - h/R = 0$; ○, $K - h/R = 0.1$; △, $K - h/R = 0.2$; and ◇, $K - h/R = 0.3$.

direction. The agreement, as expected, is quite good with the original mesh described earlier. The VABS results do not exhibit a strictly linear behavior because of the presence of some higher-order terms in k_α coming from $1/\sqrt{g}$.

Note that to get similar accuracy for the coupling term when the initial curvature is about the x_2 direction, a more refined mesh has to be used. The rate of convergence depends much more strongly on the number of elements used along the x_2 direction, with a little influence from k_2, indicating that special attention should be given to the mesh when dealing with initial curvature for a thin strip. This slow convergence takes place with the coupling term, although the main diagonal terms of the stiffness matrix converge quickly and without difficulty.

Composite Test

Consider, again, a thin rectangular cross section with dimensions $2a_2 = 26.8$ mm and $2a_3 = 2.68$ mm. Two layups were studied, following Ref. 5, and both were discretized using a 12×20 six-node-element mesh (quadratic interpolation across the width). The first is a symmetric layup $[\theta/-\theta]_s$, where θ is the ply angle, and the second is an antisymmetric configuration $[\theta_2/-\theta_2]_T$. The material used is T300/5208 graphite/epoxy,[5] the properties of which are given in Table 1.

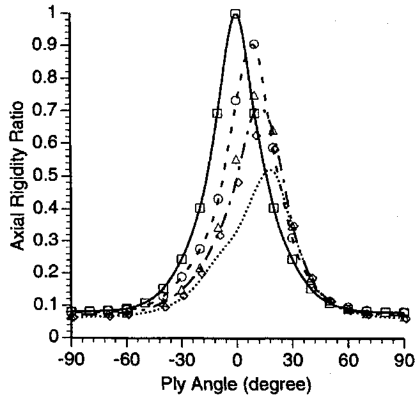

Fig. 8 Axial rigidity ratio of a $[\theta_2/-\theta_2]_T$ composite thin-strip beam for different levels of initial twist: ——, VABS − h/R = 0; - - -, VABS − h/R = 0.1; — -, VABS − h/R = 0.2; ...., VABS − h/R = 0.3; □, $K - h/R = 0$; ○, $K - h/R = 0.1$; △, $K - h/R = 0.2$; and ◇, $K - h/R = 0.3$.

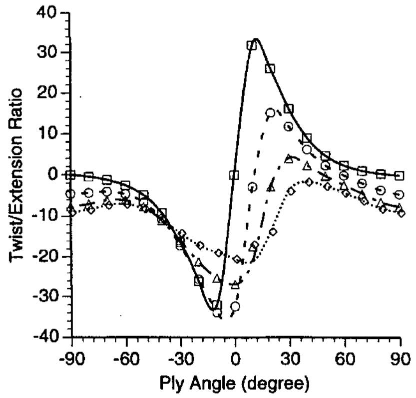

Fig. 9 Twist/extension ratio of a $[\theta_2/-\theta_2]_T$ composite thin-strip beam for different levels of initial twist: ——, VABS−h/R = 0; - - -, VABS −h/R = 0.1; — -, VABS −h/R = 0.2; ...., VABS −h/R = 0.3; □, $K - h/R = 0$; ○, $K - h/R = 0.1$; △, $K - h/R = 0.2$; and ◇, $K - h/R = 0.3$.

Initially Twisted Case

We compare VABS results with the numerical results of Kosmatka[5] for pretwisted-beam stiffness/flexibility constants. Nondimensionalized cross-sectional constants were used to set a common basis for comparision. Let's introduce the following definitions:

$$\text{Torsional rigidity ratio} = \frac{F^0_{cl(2,2)}}{F^{rr}_{cl(2,2)}}$$

$$\text{Axial rigidity ratio} = \frac{F^0_{cl(1,1)}}{F^{rr}_{cl(1,1)}}$$

$$\text{Nondimensional extension-twist coupling} = F^{rr}_{cl(1,2)} E_{LL} a_2^3$$

$$\text{Twist/extension ratio} = \frac{F^{rr}_{cl(1,2)} a_2}{F^{rr}_{cl(1,1)}}$$

where F^0_{cl} is the flexibility matrix for a prismatic unidirectional ($\theta = 0$ deg) thin strip. Figures 6–9 describe cross-sectional constants related to the antisymmetric layup and Figs. 10–13 describe those related to the symmetric layup, each with respect to the ply angle and the level of nondimensional pretwist (h/R). In these plots, the symbols represent the results from Ref. 5 (abbreviated by K) and the present results by lines (VABS). Note that the basic trends are the same between the two numerical methods.

Initially Curved Case

For the preliminary study of the influence of initial curvature in the stiffness constants of a composite beam, we kept the same thin strip used in the pretwisted case. The initial curvature (again nondimensionalized by h/R) was taken about x_3 direction, so that $k_3 \geq 0$. As mentioned before, for this case there seem to be no other results available for comparison with the present ones, to the authors' knowledge.

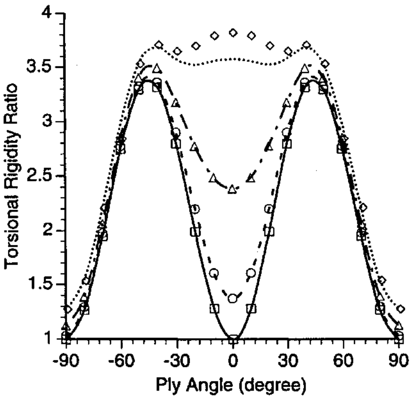

Fig. 10 Torsional rigidity ratio of a $[\theta/-\theta]_s$ composite thin-strip beam for different levels of initial twist: ——, VABS – $h/R = 0$; - - -, VABS – $h/R = 0.1$; —-, VABS – $h/R = 0.2$; ...., VABS – $h/R = 0.3$; □, $K - h/R = 0$; ○, $K - h/R = 0.1$; △, $K - h/R = 0.2$; and ◇, $K - h/R = 0.3$.

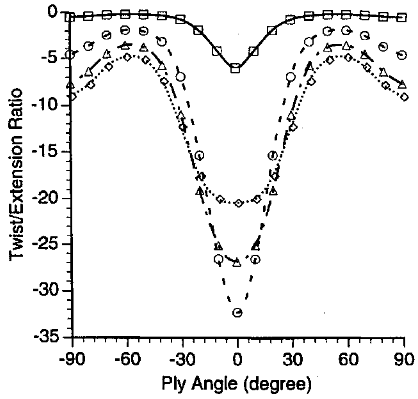

Fig. 13 Twist/extension ratio of a $[\theta/-\theta]_s$ composite thin-strip beam for different levels of initial twist: ——, VABS – $h/R = 0$; - - -0, VABS – $h/R = 0.1$; —-, VABS – $h/R = 0.2$; ...., VABS – $h/R = 0.3$; □, $K - h/R = 0$; ○, $K - h/R = 0.1$; △, $K - h/R = 0.2$; and ◇, $K - h/R = 0.3$.

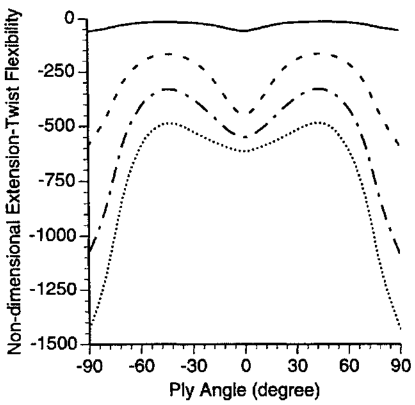

Fig. 11 Extension-twist coupling flexibility of a $[\theta/-\theta]_s$ composite thin-strip beam for different levels of initial twist: ——, VABS – $h/R = 0.01$; - - -, VABS – $h/R = 0.1$; —-, VABS – $h/R = 0.2$; and ...., VABS – $h/R = 0.3$.

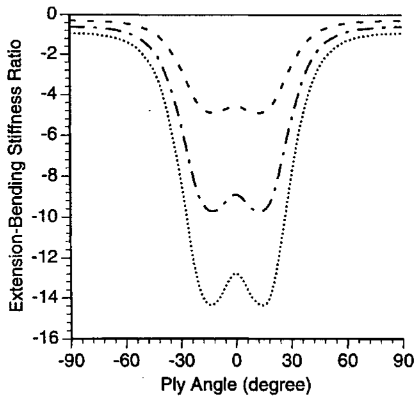

Fig. 14 Extension-bending coupling stiffness ratio of a $[\theta_2/-\theta_2]_T$ composite thin-strip beam for different levels of initial k_3: ——, VABS – $h/R = 0.01$; - - -, VABS – $h/R = 0.1$; —-, VABS – $h/R = 0.2$; and ...., VABS – $h/R = 0.3$.

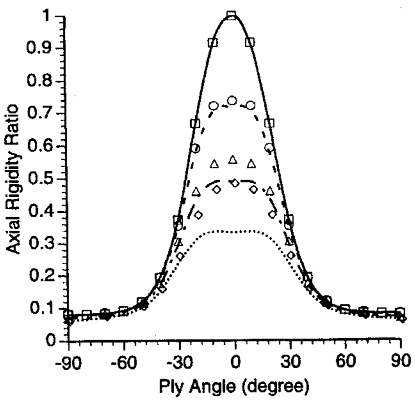

Fig. 12 Axial rigidity ratio of a $[\theta/-\theta]_s$ composite thin-strip beam for different levels of initial twist: ——, VABS – $h/R = 0$; - - -, VABS –$h/R = 0.1$; —-, VABS – $h/R = 0.2$; ...., VABS – $h/R = 0.3$; □, $K - h/R = 0$; ○, $K - h/R = 0.1$; △, $K - h/R = 0.2$; and ◇, $K - h/R = 0.3$.

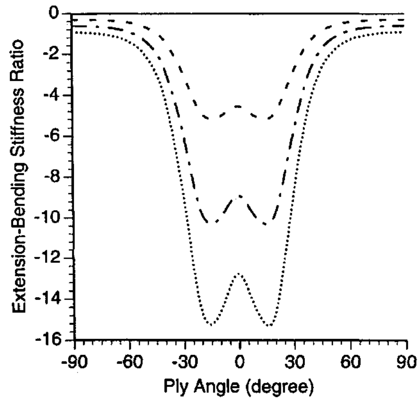

Fig. 15 Extension-bending coupling stiffness ratio of a $[\theta/-\theta]_s$ composite thin-strip beam for different levels of initial curvature (k_3): ——, VABS – $h/R = 0.01$; - - -, VABS – $h/R = 0.1$; —-, VABS – $h/R = 0.2$; and ...., VABS – $h/R = 0.3$.

Figures 14 and 15 show the variation of the extension-bending stiffness constants for the $[\theta_2/\theta_2]_T$ and $[\theta/\theta]_s$ configurations, respectively. As one can see, the trends are very similar, apparently dominated by the geometric effect of the k_3 curvature. The maximum values of the coupling terms occur for $\theta \approx \pm 15$ deg. The effects of the symmetric or antisymmetric layups are not significant for the general behavior. This is also the case for all of the other constants. In fact, the effective axial rigidity as well as the effective bending rigidity $[F^{rr}_{cl(3,3)}]$ are insensitive for the variation of k_3. Both effective torsional rigidity and the other effective bending rigidity $[F^{rr}_{cl(4,4)}]$ exhibit similar results, with the difference that there is an influence of initial curvature of up to 10% localized around $\theta = \pm 45$ and 0 deg, respectively.

Concluding Remarks

In an effort to generalize previously published work in dimensional reduction analysis of initially curved and twisted beams made of nonhomogeneous, anisotropic materials, an asymptotically exact methodology, based on geometrically nonlinear, three-dimensional elasticity, has been developed. The analysis is subject only to the restrictions that the strain is small compared with unity and that the maximum dimension of the cross section is small relative to the wavelength of deformation along the beam and to the minimum radius of initial curvature and/or twist. The final one-dimensional strain energy per unit length exhibits asymptotically correct second-order dependence of the initial curvature and twist parameters. The sectional constants are obtained by a finite element discretization over the cross-sectional plane.

Numerical results obtained for both isotropic and composite thin strips are presented and compared with available published results from Knowles and Reissner,[20] Chu,[21] Rosen,[22] Berdichevsky and Staroselsky,[23] and Kosmatka.[5] These cases include initially twisted straight beams. Results are also presented for untwisted curved beams, although other than the first-order corrections available in the work of Berdichevsky and Staroselsky,[23] there appear to be no other results available in the literature for direct comparison of stiffnesses. All of the previously published analyses are inherently restricted to special cases of the present one, and the asymptotical correctness of some of those analyses is hard to determine. The present theory is the only one that has a framework such that all of the following is achieved.

1) It is derived from geometrically nonlinear, three-dimensional elasticity, and all of the effects are captured in a consistent way (no ad hoc assumptions).

2) The two-dimensional linear sectional analysis and the one-dimensional nonlinear beam analysis both stem from the same formulation.

3) It is applicable to general cross-sectional geometries and material distributions.

4) The framework allows for inclusion of nonclassical beam degrees of freedom (see, for example, Ref. 17).

Plans for future work include further validation for initially curved beams, including comparisons that involve global deformation (i.e., one-dimensional results, such as free-vibration characteristics, that make use of the present cross-sectional properties).

Acknowledgments

This work was supported by the Center of Excellence for Rotorcraft Technology at the Georgia Institute of Technology, sponsored by the U.S. Army Research Office. The technical monitor was Gary L. Anderson.

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